

Rigorous 3D Integrated Adjustment of GPS Baselines, Geodetic Total Station and Levelling Measurements

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Key words: integrated adjustment, GPS, geodetic total station, spirit levelling, deflection of vertical, geoid undulation, bank failure

SUMMARY

In the first phase of our bank failure monitoring GPS baselines were controlled by precise levelling. After the approximately 8 m fast subsidence, geodetic total station had to be applied to maintain the continuity of the 3D network measurements.

Therefore new procedure was developed where the integrated measurements are adjusted in GPS reference frame. The available deflections of the vertical and the geoid undulations can be treated as measurements. Quasi linear observation equations are introduced for geodetic total station measurements. This is more advantageous than the usual procedure.

The method and the effectiveness of the 3D integrated adjustment will be demonstrated using the practical data measured during the bank failure monitoring of Danube in Dunaszekcso, Hungary.

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1. INTRODUCTION

The first phase of the bank failure investigations in Dunaszekcső, Hungary was summarized in Ujvari et al. (2009). During the continuous monitoring all together five deeply founded reinforced concrete pillars were established which can be handled as stable reference benchmarks. Additionally 32 surface monitoring benchmarks were placed at the two sides of the subsiding areas.

Redundant GPS base lines were measured in all the observation epochs. The baselines were adjusted by the procedure describe in Banyai (1991, 2005). Based on the generalized free network approach the coordinate changes of the pillars are minimized with respect to the first epoch and the other points are allowed to change freely. The stability of the pillars is investigated by statistical tests.

The monitoring benchmarks were connected by spirit levelling to improve the accuracy of the relative height changes measured by GPS receivers. The next equation was used

$$h_i = \frac{\sum h_i^e}{n} + (H_i - \frac{\sum H_i}{n}) , \quad (1)$$

where h^e is the height above the ellipsoid determined by GPS, H is the height determined by spirit levelling and n is the number of monitoring points. It means that the average height is determined by GPS while the height differences are determined by spirit levelling. This is a simple but not rigorous solution, where the geoid undulations were neglected because of the small area.

After the approximately 8 m fast subsidence (February 17, 2008) three monitoring point disappeared, not all the monitoring points can be connected by spirit levelling and additionally at eight points – near to the new edge – proper GPS measurement cannot be carried out any longer (Fig. 1). Therefore, to maintain the continuity of the 3D network observations, geodetic total station (GTS) measurements have to be applied. Rigorous 3D integrated adjustment procedure was developed, which will be demonstrated in this paper.

2. BASIC PRINCIPLES OF INTEGRATED ADJUSTMENT

2.1 Observations

The GPS derived baseline components are interpreted in Earth Centered and Earth Fixed (ECEF) coordinate systems, while the GTS (and theodolite) measurements are interpreted in geographical (or astronomical) systems. The differences according to (Bomford 1980)



Fig 1. Aerial photo of the subsiding area

$$\begin{aligned}
 \zeta &= (\varphi^* - \varphi) \\
 \eta &= (\lambda^* - \lambda) \cos \varphi , \\
 u &= h - H
 \end{aligned}
 \tag{2}$$

where ζ and η are the deflections of the vertical, u is the geoid undulation, the ellipsoidal coordinates in the ECEF system are φ , λ and h , moreover the astronomical (or geographic) coordinates are φ^* , λ^* and H is the orthometric height above the geoid.

The GTS measurement are carried out in the astronomical topocentric system – attached to the plumb line – that can be transformed to the ellipsoidal topocentric system (Vanicek and Krakiwsky 1986)

$$\begin{aligned}
 \alpha &= \alpha^* - (\zeta \sin \alpha - \eta \cos \alpha) \cot \zeta , \\
 \varsigma &= \varsigma^* + (\zeta \sin \alpha + \eta \cos \alpha)
 \end{aligned}
 \tag{3}$$

where α and ς are the azimuth and zenith angles in the ellipsoidal, α^* and ς^* are the azimuth and zenith angles in the astronomical systems. Instead of azimuths only directions (d) can be directly measured, which are different from the azimuth with the orientation unknowns

$$\begin{aligned}
 d^* &= \alpha^* - \omega^* \\
 d &= \alpha - \omega
 \end{aligned}
 \tag{4}$$

In the case of one direction the following equations can be written

$$\begin{bmatrix} d_{PQ} \\ \zeta_{PQ} \\ S_{PQ} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\sin\alpha_{PQ} \cot\zeta_{PQ} & \cos\alpha_{PQ} \cot\zeta_{PQ} \\ 0 & 1 & 0 & \sin\alpha_{PQ} & \cos\alpha_{PQ} \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} d_{PQ}^* \\ \zeta_{PQ}^* \\ S_{PQ} \\ \xi_P \\ \eta_P \end{bmatrix}, \quad (5)$$

$$l_c = B \cdot l$$

where S_{PQ} is the slant distance and l_c is the vector of corrected measurements. Because of orientation unknowns at least two directions have to be measured from one GTS standing point. Every standing point has two additional deflection components. The α and ζ quantities are computed from the preliminary coordinates.

The effect of the deflections of the vertical can be neglected on the standing heights of the GTS instruments and reflecting prisms, furthermore on the readings on the levelling rods.

According to eq. (2) in the case of one levelled height difference the following equations can be written

$$(h_p - h_q) = (H_p - H_q) + (u_p - u_q)$$

$$\Delta h_{PQ} = [1 \quad 1 \quad -1] \cdot \begin{bmatrix} \Delta H_{PQ} \\ u_p \\ u_q \end{bmatrix}, \quad (6)$$

$$l_c = B \cdot l$$

where ΔH_{PQ} is the levelled height difference an l_c is the corrected corresponding ellipsoidal height difference.

In the case of GPS baselines, which are estimated with respect to the base station, it can be supposed that they may be differentially rotated about the base station and the coordinate axes (Banyai 1991). In the case of one baseline

$$\begin{bmatrix} \Delta X_{PQ} \\ \Delta Y_{PQ} \\ \Delta Z_{PQ} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\Delta Z_{PQ}^0 / \rho & \Delta Y_{PQ}^0 / \rho \\ 0 & 1 & 0 & \Delta Z_{PQ}^0 / \rho & 0 & -\Delta X_{PQ}^0 / \rho \\ 0 & 0 & 1 & -\Delta Y_{PQ}^0 / \rho & \Delta X_{PQ}^0 / \rho & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta X'_{PQ} \\ \Delta Y'_{PQ} \\ \Delta Z'_{PQ} \\ \alpha \\ \beta \\ \gamma \end{bmatrix}, \quad (7)$$

$$l_c = B \cdot l$$

where $\Delta X'_{PQ}$, $\Delta Y'_{PQ}$, $\Delta Z'_{PQ}$ are the GPS derived baseline components ΔX_{PQ}^0 , ΔY_{PQ}^0 , ΔZ_{PQ}^0 are computed from the preliminary coordinates, $\rho = 206264.8$ arc sec, α , β , γ are the rotations about X,Y,Z axes in arc sec and l_c is the vector of corrected measurements. Every baseline – belonging to one base station which are measured and adjusted in one session – has three additional rotation parameters.

2.2 Adjustment models

If it is reasonable, the deflections of the vertical, the geoid undulations and the rotations about the coordinate axes can be handled as real observations, too. In that case the unified least squares adjustment of observations and functionally independent parameters (Mikhail 1967) can be applied

$$\begin{aligned}
 \mathbf{B}\mathbf{v} &= \mathbf{A}\mathbf{x} - \mathbf{b} \\
 \mathbf{P} &= \mathbf{Q}^{-1} \\
 \mathbf{P}_c &= (\mathbf{B}\mathbf{Q}\mathbf{B}')^{-1} \\
 \hat{\mathbf{x}} &= (\mathbf{A}'\mathbf{P}_c\mathbf{A})^{-1}(\mathbf{A}'\mathbf{P}_c\mathbf{b}) \\
 \hat{\mathbf{v}} &= \mathbf{Q}\mathbf{B}'\mathbf{P}_c(\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}) \\
 \mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} &= (\mathbf{A}'\mathbf{P}_c\mathbf{A})^{-1} = \mathbf{N}^{-1} \\
 \mathbf{Q}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} &= \mathbf{Q}\mathbf{B}'(\mathbf{P}_c - \mathbf{P}_c\mathbf{A}\mathbf{N}^{-1}\mathbf{A}'\mathbf{P}_c)\mathbf{B}\mathbf{Q} \\
 \mathbf{Q}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} &= \mathbf{Q} - \mathbf{Q}_{\hat{\mathbf{v}}\hat{\mathbf{v}}} \\
 \hat{\sigma}_0^2 &= \frac{\mathbf{v}'\mathbf{P}\mathbf{v}}{m-n}
 \end{aligned} \tag{8}$$

where

- \mathbf{x} is an unknown correction vector of the preliminary parameters,
- $\hat{\mathbf{x}}$ is the the estimated vector of unknown corrections,
- \mathbf{v} is the unknown residual vector of the observations,
- $\hat{\mathbf{v}}$ is the the estimated vector of unknown residuals,
- \mathbf{B} is a coefficient matrix of the observations,
- \mathbf{A} is a coefficient matrix of the parameters,
- \mathbf{b} is a misclosure vector of the corrected measurements and the preliminary parameters,
- \mathbf{Q} is a weight-coefficient matrix of the observations,
- \mathbf{P} is a weight matrix of the observations,
- $\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}$, $\mathbf{Q}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}$ and $\mathbf{Q}_{\hat{\mathbf{v}}\hat{\mathbf{v}}}$ are the weight-coefficient matrices of the estimated corrections, residuals and measurements respectively,
- $\hat{\sigma}_0^2$ is the estimated standard deviation of unit weight, m is a number of measurements and n is a number of estimated parameters.

The observation equations have to be solved iteratively; therefore, it is reasonable to solve the corrected observations in the iterative process. This approach reduces eqs. (8) to the adjustment of the indirect observations (Mikhail 1967)

$$\begin{aligned}
\mathbf{v}_c &= \mathbf{A} \mathbf{x} - \mathbf{b} \\
\mathbf{Q}_c &= \mathbf{B} \mathbf{Q} \mathbf{B}^t \\
\mathbf{P}_c &= \mathbf{Q}_c^{-1} \\
\hat{\mathbf{x}} &= (\mathbf{A}^t \mathbf{P}_c \mathbf{A})^{-1} (\mathbf{A}^t \mathbf{P}_c \mathbf{b}) \\
\hat{\mathbf{v}}_c &= \mathbf{A} \hat{\mathbf{x}} - \mathbf{b} \\
\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}} &= (\mathbf{A}^t \mathbf{P}_c \mathbf{A})^{-1} = \mathbf{N}^{-1} \\
\mathbf{Q}_{\hat{\mathbf{v}}_c \hat{\mathbf{v}}_c} &= \mathbf{Q}_c - \mathbf{A} \mathbf{N}^{-1} \mathbf{A}^t \\
\mathbf{Q}_{\hat{\mathbf{i}}_c \hat{\mathbf{i}}_c} &= \mathbf{Q}_c - \mathbf{Q}_{\hat{\mathbf{v}}_c \hat{\mathbf{v}}_c} \\
\hat{\sigma}_0^2 &= \frac{\mathbf{v}_c^t \mathbf{P} \mathbf{v}_c}{m - n}
\end{aligned} \tag{9}$$

Eq. (9) leads to the same results; however the original observations can be estimated only by eq. (8) in the last step of the adjustment.

2.3 Observation equations

The corrected GTS measurements are not linear functions of the Cartesian coordinates; therefore – in the case of eq. (8) – the coefficient matrix \mathbf{A} contains the derivatives with respect to these coordinates. These derivatives can be found e.g. in Leick (1995) or in Strang and Borre (1997). This approach requires very good preliminary coordinates, therefore in the iterative phase a new concept is applied. According to Strang és Borre (1997) the topocentric coordinate differences are derived

$$\begin{aligned}
\Delta X_{PQ}^e &= S_{PQ} \cdot \sin \zeta_{PQ} \cdot \cos(\omega_P + d_{PQ}) \\
\Delta Y_{PQ}^e &= S_{PQ} \cdot \sin \zeta_{PQ} \cdot \sin(\omega_P + d_{PQ}) \quad , \\
\Delta Z_{PQ}^e &= S_{PQ} \cdot \cos \zeta_{PQ}
\end{aligned} \tag{10}$$

which can be transformed to the ECEF system

$$\begin{aligned}
\Delta X_{PQ} &= -\Delta X_{PQ}^e \cdot \sin \lambda_P - \Delta Y_{PQ}^e \cdot \sin \varphi_P \cdot \cos \lambda_P + \Delta Z_{PQ}^e \cdot \cos \varphi_P \cdot \cos \lambda_P \\
\Delta Y_{PQ} &= \Delta X_{PQ}^e \cdot \cos \lambda_P - \Delta Y_{PQ}^e \cdot \sin \varphi_P \cdot \sin \lambda_P + \Delta Z_{PQ}^e \cdot \cos \varphi_P \cdot \sin \lambda_P \quad . \\
\Delta Z_{PQ} &= \quad \quad \quad + \Delta Y_{PQ}^e \cdot \cos \varphi_P \quad \quad \quad + \Delta Z_{PQ}^e \cdot \sin \varphi_P
\end{aligned} \tag{11}$$

For the computation of eqs. (5), (10) and (11) the ellipsoidal coordinates, zenith angles, azimuths and – consequently – the preliminary orientation unknowns can be computed from the preliminary Cartesian coordinates.

According to eq. (9) this procedure leads to the quasy-linear observation equations

$$\begin{bmatrix} v_{\Delta X} \\ v_{\Delta Y} \\ v_{\Delta Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & \partial \Delta X_{PQ} / \partial \omega_P \\ 0 & 1 & 0 & 0 & -1 & 0 & \partial \Delta Y_{PQ} / \partial \omega_P \\ 0 & 0 & 1 & 0 & 0 & -1 & \partial \Delta Z_{PQ} / \partial \omega_P \end{bmatrix} \cdot \begin{bmatrix} x_Q \\ y_Q \\ z_Q \\ x_P \\ y_P \\ z_P \\ \omega_P \end{bmatrix}, \quad (12)$$

$$- \begin{bmatrix} \Delta X_{PQ} - X_Q^0 + X_P^0 - m_Q \cdot \cos \varphi_Q^0 \cdot \cos \lambda_Q^0 + m_P \cdot \cos \varphi_P^0 \cdot \cos \lambda_P^0 \\ \Delta Y_{PQ} - Y_Q^0 + Y_P^0 - m_Q \cdot \cos \varphi_Q^0 \cdot \sin \lambda_Q^0 + m_P \cdot \cos \varphi_P^0 \cdot \sin \lambda_P^0 \\ \Delta Z_{PQ} - Z_Q^0 + Z_P^0 - m_Q \cdot \sin \varphi_Q^0 + m_P \cdot \sin \varphi_P^0 \end{bmatrix}$$

where the preliminary values are signed by 0 upper indices, m_P and m_Q are the standing heights of the GTS instrument and the prism above the geodetic benchmarks, respectively. The coordinate differences are formally linear functions of the unknown coordinates, however, they are computed from the preliminary coordinates; consequently, they are only quasi-linear quantities.

Because the GTS measurements are transformed to the coordinate differences the Q_c matrix of eq. (9) has to be modified

$$Q_c = DBQ B' D' . \quad (13)$$

The coefficient matrix D contains the derivatives of the coordinate differences with respect to the measured directions, elevation angles and slant distances. These can be derived – together with the derivatives with respect to the orientation unknown in eq. (12) – very easily from eqs. (11) and (10) using the chain rule. The original GTS measurements and the deflections are treated as uncorrelated measurements, therefore

$$Q = \langle \sigma_d^2 / \sigma_0^2, \sigma_c^2 / \sigma_0^2, \sigma_s^2 / \sigma_0^2, \sigma_\xi^2 / \sigma_0^2, \sigma_\eta^2 / \sigma_0^2 \rangle, \quad (14)$$

where σ quantities are the a priori standard deviation of the different observations and σ_0 is the a priori standard deviation of the unit weight.

The observation equation of one height difference

$$v_{\Delta h} = \begin{bmatrix} \partial \Delta h_{PQ} / \partial X_Q & \partial \Delta h_{PQ} / \partial Y_Q & \partial \Delta h_{PQ} / \partial Z_Q \\ \partial \Delta h_{PQ} / \partial X_P & \partial \Delta h_{PQ} / \partial Y_P & \partial \Delta h_{PQ} / \partial Z_P \end{bmatrix} \cdot \begin{bmatrix} x_Q \\ y_Q \\ z_Q \\ x_P \\ y_P \\ z_P \end{bmatrix}, \quad (15)$$

$$- [\Delta H_{PQ} - (h_P^0 - h_Q^0) - (u_P - u_Q)]$$

and the weight-coefficient matrices

$$\mathbf{Q} = \langle \sigma_{\Delta H}^2 / \sigma_0^2, \sigma_{u_p}^2 / \sigma_0^2, \sigma_{u_Q}^2 / \sigma_0^2 \rangle$$

$$\mathbf{Q}_c = \left[\frac{\sigma_{\Delta H}^2 + \sigma_{u_Q}^2 + \sigma_{u_p}^2}{\sigma_0^2} \right] \quad (16)$$

Based on spherical approximation the derivatives of eq. (15) are given in Strang and Borre (1997).

The corrected GPS baseline components are linear functions of coordinate unknowns

$$\begin{bmatrix} v_{\Delta X} \\ v_{\Delta Y} \\ v_{\Delta Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_Q \\ y_Q \\ z_Q \\ x_P \\ y_P \\ z_P \end{bmatrix} - \begin{bmatrix} \Delta X_{PQ} - (X_Q^0 - X_P^0) \\ \Delta Y_{PQ} - (Y_Q^0 - Y_P^0) \\ \Delta Z_{PQ} - (Z_Q^0 - Z_P^0) \end{bmatrix} \quad (17)$$

The weight-coefficient matrices are

$$\mathbf{Q} = \langle \sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2 \rangle$$

$$\mathbf{Q}_c = \frac{1}{\sigma_0^2} \mathbf{B} \begin{bmatrix} s^2 \mathbf{M} & 0 \\ 0 & \mathbf{Q} \end{bmatrix} \mathbf{B}^t \quad (18)$$

where \mathbf{M} is a variance-covariance matrix determined together with GPS baseline components and s is the arbitrary scale factor to provide the proper weight ratio of the different type of observations.

The unknown parameters of GPS baseline components can be completed by scale correction and phase center offsets (Banyai 1991, 2005), too.

3. THE ADJUSTMENT PROCEDURE

The adjustment procedure is based on the introduced basic principles. The GTS measurements and the GPS baseline components can be adjusted alone; however the levelled height differences are only additional measurements, which improve the less accurate height components of the other GTS and GPS measurements.

Depending on the area of the geodetic networks and on the available deflections of the vertical they

- can be neglected,
- can be used as corrections only,
- can be handled as zero quantities with known standard deviations or
- can be handled as real measurements (estimated by proper model together with their standard deviation).

The same can be done with the geoid undulations in the case of levelled height differences. However the measured height differences and geoid undulations are linearly dependent (eq.6),

consequently they can not be effectively separated by eq. (8). They are separated only by their weight ratios.

If there are large discrepancies amongst the GPS baselines, which cannot be handled by individual phase offsets, the additional rotations can be handled as zero quantities with known standard deviation.

The most crucial point of the integrated adjustment is the proper selection of the a priori standard deviations of the measurements. Because the standard deviations of the GPS baseline components are usually over estimated, the arbitrary scale factor (s) is introduced. In the first phase of the integrated adjustment it is reasonable to adjust the GPS baselines alone. In that case the ratio of the estimated and a priori standard deviations of the unit weights

$$s = \frac{\hat{\sigma}_0}{\sigma_0} \quad (17)$$

can be used to estimate the preliminary scale factor.

During the adjustment sufficient number of coordinates can be fixed, or according to the free network approach, the squares sum of the selected benchmark's coordinate changes can be minimized with respect to the coordinates determined in the reference epoch. The other benchmarks are allowed to change freely.

The estimated residuals are investigated by τ and χ^2 tests.

4. EXPERIENCES OF PRACTICAL APPLICATION

The introduced procedure was tested thoroughly during the practical applications described in the introductory chapter.

The introduction of the quasy-linear observation equations of GTS measurements proved to be very useful. If the GTS derived benchmarks are connected to at least two GPS derived benchmarks, their preliminary coordinates can be chosen arbitrary. In that case the traditional approach cannot be used.

During the integrated adjustments the preliminary GPS scale factors varied usually between 10 and 25, which describes the quality of the applied GPS measurements. No rotations of the baselines were taken into account.

The a priori standard deviations of the slant distances (σ_s) are treated by 1mm+1ppm values. The a priori standard deviations of the direction and zenith angels are chosen to be equivalent with σ_s in the distance of the prisms. Because of the small area the deflections of the vertical was treated as zero quantities with 1.0 arc sec standard deviations.

The leveled height differences were treated by 0.15-0.30 mm standard deviations and the geoid undulations ware neglected.

The additional levelling measurements significantly improved the height components of the estimated coordinates, and the detection of the errors in the standing height measurements and the vertical GPS phase center offsets.

The estimated average standard deviations of the coordinates varied between 0.2-0.5 mm.

The coordinate changes of the benchmark 2003 and 4002 (Fig. 1), which were determined only by GTS measurement, are given in Figs. 2 and 4. These show the effectiveness of the integrated adjustments and the slow post event movements of the collapsed areas.

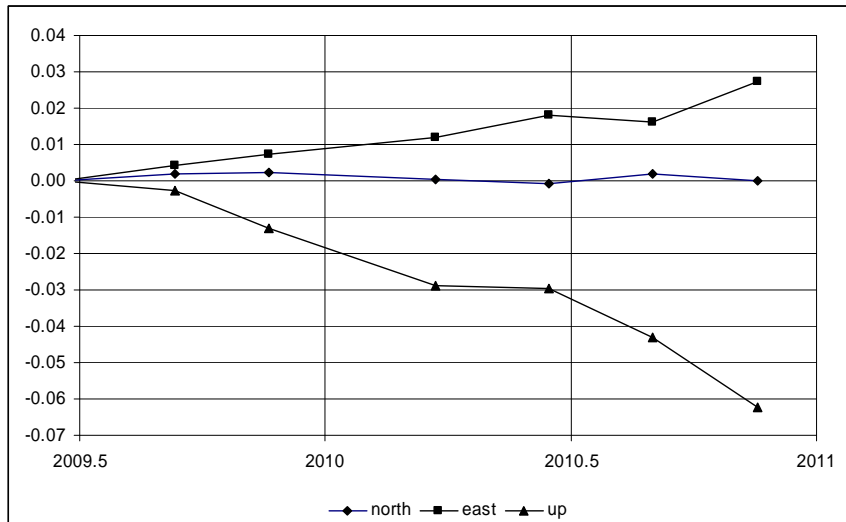


Fig 2. Coordinate changes of benchmark 2003 in meter

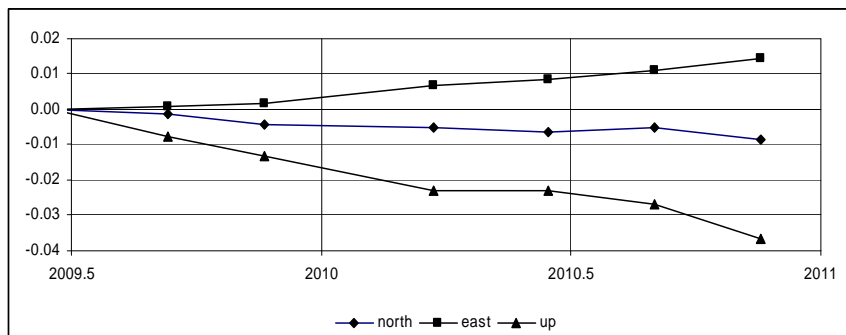


Fig 3. Coordinate changes of benchmark 4002 in meter

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BIOGRAPHICAL NOTES

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